Problem Set: The Romer Endogenous Growth Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Time is discrete. There is no population growth:

$$\bar{L}_t = \bar{L}, \qquad g_L = 0.$$

The economy has two labour uses: final-goods production and R&D,

$$L_{y,t} + L_{a,t} = \bar{L}, \qquad L_{a,t} = \bar{\ell}\bar{L}, \qquad L_{y,t} = (1-\bar{\ell})\bar{L}, \qquad 0 < \bar{\ell} < 1.$$

The final-good sector and idea-production sector are

$$Y_t = A_t K_t^{\alpha} L_{y,t}^{1-\alpha}, \qquad 0 < \alpha < 1,$$

$$A_{t+1} - A_t = \bar{z} A_t L_{a,t}, \qquad \bar{z} > 0.$$

Capital accumulation is

$$K_{t+1} - K_t = sY_t - \delta K_t, \qquad 0 < s < 1, \quad 0 < \delta < 1.$$

Question 1: Nonrival Ideas, Increasing Returns, and Balanced Growth [Total: 60 marks]

(a) Holding A_t fixed, prove that final-good production has constant returns to scale in rival inputs $(K_t, L_{y,t})$. Then show that if ideas are scaled together with rival inputs, production displays increasing returns:

$$F(2K_t, 2L_{y,t}, 2A_t) > 2F(K_t, L_{y,t}, A_t).$$

Explain the role of nonrivalry of ideas.

(b) Derive output per person,

$$y_t \equiv \frac{Y_t}{\bar{L}},$$

as a function of A_t, $K_t/\bar{L}$, and $\bar{\ell}$. Show that a higher stock of ideas raises every person's output because ideas are nonrival.

(c) Derive the growth rate of ideas,

$$g_A \equiv \frac{A_{t+1} - A_t}{A_t}.$$

Then derive g_Y from the production function using growth-rate algebra. Explain why $g_{L_y} = 0$.

(d) Use capital accumulation to show

$$g_K = s\frac{Y_t}{K_t} - \delta.$$

Along a balanced-growth path, Y_t/K_t is constant. Use this to prove that $g_K = g_Y$ and derive

$$\boxed{g_Y = g_K = g_y = \frac{\bar{z}\bar{\ell}\bar{L}}{1-\alpha}}.$$

(e) Compute g_A, g_Y, and g_y when

$$\alpha = \frac{1}{3}, \qquad \bar{z} = 0.0005, \qquad \bar{\ell} = 0.04, \qquad \bar{L} = 1000.$$

Then interpret the "scale effect" in this model.

Question 2: Steady-State Ratios, Labour Allocation, and Policy Trade-offs [Total: 40 marks]

(a) Use the balanced-growth condition and capital accumulation to derive the steady-state output-capital ratio:

$$\frac{Y^*}{K^*} = \frac{g_K + \delta}{s}.$$

Then derive steady-state capital per person:

$$k_t^* \equiv \frac{K_t^*}{\bar{L}} = \left(\frac{s}{g_K + \delta}\right)^{1/(1-\alpha)} A_t^{1/(1-\alpha)}(1-\bar{\ell}).$$

(b) Derive steady-state output per person:

$$y_t^* = \left(\frac{s}{g_K + \delta}\right)^{\alpha/(1-\alpha)} A_t^{1/(1-\alpha)}(1-\bar{\ell}).$$

Explain why y_t^* grows even though population is constant.

(c) Derive the effect of $\bar{\ell}$ on the long-run growth rate g_y. Then discuss the static cost of increasing $\bar{\ell}$ through the term $(1-\bar{\ell})$ in y_t^*. Interpret the intertemporal trade-off.

(d) Suppose $\alpha = 1/3$, $s = 0.24$, $\delta = 0.05$, $\bar{z} = 0.0005$, $\bar{L} = 1000$, and $\bar{\ell} = 0.04$. Compute g_K and Y^*/K^*. Then derive the general comparative statics of g_y with respect to $\bar{z}$, $\bar{\ell}$, $\bar{L}$, and α.